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Gram Polynomials and the Kummer Function

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Let $\{\phi_k\}_{k=0}^{n}$, $n<m$, be a family of polynomials orthogonal with respect to the positive semi-definite bilinear form

$$(g, h)_d := \frac{1}{m} \sum_{j=1}^{m} g(x_j)\, h(x_j), \qquad x_j := -1 + (2j-1)/m.$$

These polynomials are known as Gram polynomials. The present paper investigates the growth of $|\phi_k(x)|$ as a function of k and m for fixed $x \in [-1, 1]$. We show that when $n \leqslant 2.5m^{1/2}$, the polynomials in the family $\{\phi_k\}_{k=0}^{n}$ are of modest size on $[-1, 1]$, and they are therefore well suited for the approximation of functions on

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this interval. We also demonstrate that if the degree k is close to m, and $m \geqslant 10$, then $\phi_k(x)$ oscillates with large amplitude for values of x near the endpoints of $[-1, 1]$, and this behavior makes ϕ_k poorly suited for the approximation of functions on $[-1, 1]$. We study the growth properties of $|\phi_k(x)|$ by deriving a second order differential equation, one solution of which exposes the growth. The connection between Gram polynomials and this solution to the differential equation suggested what became a long-standing conjectured inequality for the confluent hypergeometric function ${}_1F_1$, also known as Kummer's function, i.e., that ${}_1F_1((1-a)/2, 1, t^2) \leqslant {}_1F_1(1/2, 1, t^2)$ for all $a \geqslant 0$. In this paper we completely resolve this conjecture by verifying a generalization of the conjectured inequality with sharp constants.

1. INTRODUCTION

Let f be a smooth function defined on the closed interval $[-1, 1]$ and assume that f is explicitly known only at the m equidistant points

$$x_k := -1 + (2k-1)/m, \qquad 1 \leqslant k \leqslant m. \tag{1}$$

We wish to approximate f on $[-1, 1]$ by a polynomial of degree n, where $n < m$. Introduce the positive semi-definite bilinear form

$$(g, h)_d := \frac{1}{m} \sum_{k=1}^{m} g(x_k)\, h(x_k) \tag{2}$$

for functions f, g continuous on $[-1, 1]$, and define the associated discrete semi-norm

$$\|g\|_d := (g, g)_d^{1/2}. \tag{3}$$

Let $\{\phi_k\}_{k=0}^{m-1}$ be the family of polynomials that are orthogonal with respect to the bilinear form (2), have positive leading coefficient and are normalized so that $\|\phi_k\|_d = 1$. The ϕ_k are known as *Gram polynomials*. These polynomials are discussed, e.g., by Dahlquist and Björck [7, Sect. 4.4.4], Hildebrand [12, Sects. 7.13 and 7.16], and Szegö [20, Sect. 2.8].

Let Π_n denote the set of all polynomials of degree at most n. The polynomial $\Phi_n \in \Pi_n$, that solves the discrete least-squares approximation problem

$$\|f - \Phi_n\|_d = \min_{\Phi \in \Pi_n} \|f - \Phi\|_d, \tag{4}$$

is given by

$$\Phi_n(x) := \sum_{k=0}^{n} \beta_k \phi_k(x), \qquad \beta_k := (\phi_k, f)_d \tag{5}$$

and is therefore simple to compute. It is the purpose of the present paper to investigate the conditions on n under which the solution Φ_n of (4) also approximates f well with respect to the uniform norm

$$\|g\|_\infty := \sup_{x \in [-1, 1]} |g(x)|.$$

In order to gain some insight into the behavior of $\|f - \Phi_n\|_\infty$, we first review two special cases: $n \ll m$ and $n = m-1$. We begin with the former. Let $\{p_k\}_{k=0}^n$ denote the *Legendre polynomials* normalized so that $\|p_k\| = 1$, where we define

$$\langle g, h \rangle := \tfrac{1}{2} \int_{-1}^{1} g(x)\, h(x)\, dx, \tag{6}$$

$$\|g\| := \langle g, g \rangle^{1/2} \tag{7}$$

for all square integrable functions on $[-1, 1]$. Analogously to (5), the solution $P_n \in \Pi_n$ of the (continuous) least-squares problem

$$\|f - P_n\| = \min_{P \in \Pi_n} \|f - P\|$$

can be written as

$$P_n(x) = \sum_{k=0}^{n} \langle p_k, f \rangle\, p_k(x).$$

In [6, p. 345] Brass proved the following result.

THEOREM 1.1. *Let $d\sigma$ be a distribution on $[-1, 1]$, and let $\{q_k\}_{k=0}^{n+1}$ be a family of orthogonal polynomials with respect to $d\sigma$. Assume the normalization $\int_{-1}^{1} q_k^2(x)\, d\sigma(x) = 1$. Let $d\sigma$ be such that*

(i) $\int_{-1}^{1} f(x)\, d\sigma(x) = \int_{-1}^{1} f(-x)\, d\sigma(x)$ *for any* $f \in C[-1, 1]$,

(ii) $\|q_k\|_\infty = q_k(1)$, $k = 0, 1, ..., n+1$.

Assume that $f \in C^{n+1}[-1, 1]$, and let $\eta_k := \int_{-1}^{1} f(x)\, q_k(x)\, d\sigma(x)$. Then

$$\left\| f - \sum_{k=0}^{n} \eta_k q_k \right\|_\infty \leqslant \frac{\|q_{n+1}\|_\infty}{\|q_{n+1}^{(n+1)}\|_\infty} \|f^{(n+1)}\|_\infty.$$

Sharpness follows by letting $f = q_{n+1}$.

We apply this result and use the known properties of the Legendre polynomials, including the fact that $\|p_k\|_\infty = p_k(1)$, to obtain

$$\|f - P_n\|_\infty \leqslant \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \|p_{n+1}\|_\infty \lim_{x \to \infty} (x^{n+1}/p_{n+1}(x)). \tag{8}$$

This inequality is used in the proof of the following bound.

PROPOSITION 1.2. *Assume that $m > n$, and let Φ_n be given by (5). Then, for $f \in C^{n+1}[-1, 1]$,*

$$\|f - \Phi_n\|_\infty \leqslant \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!} \cdot \frac{\pi^{1/2}}{2} n^{1/2}(1 + O(n^{-1})) + \hat{c}_n O(m^{-2}), \tag{9}$$

where the $O(n^{-1})$-term is independent of m and the $O(m^{-2})$-term is independent of n. The constant $\hat{c}_n$ is independent of f and m.

Proof. The bilinear form (2) corresponds to a discretization by the rectangle rule of (6), which has a discretization error $O(m^{-2})$. Therefore, there are constants c_k, such that for each k,

$$\phi_k(x) = p_k(x) + c_k O(m^{-2}), \qquad m \to \infty, \tag{10}$$

uniformly for $x \in [-1, 1]$; see Wilson [21] for details. It follows from (10) that there are constants $\hat{c}_n$, such that for each n,

$$\begin{aligned} \|f - \Phi_n\|_\infty &\leqslant \|f - P_n\|_\infty + \|P_n - \Phi_n\|_\infty \\ &= \|f - P_n\|_\infty + \hat{c}_n O(m^{-2}), \qquad m \to \infty. \end{aligned} \tag{11}$$

Substitute (8) into (11) and use the following equalities that follow from results in [20, Sect. 4.7],

$$\|p_{n+1}\|_\infty = (2n+3)^{1/2}, \tag{12}$$

$$\lim_{x \to \infty} (x^{n+1}/p_{n+1}(x)) = 2^n \binom{2n+1}{n}^{-1} (2n+3)^{-1/2}, \tag{13}$$

and apply Stirling's formula to bound the binomial coefficient in (13). This shows the proposition. ∎

Let $Q_n \in \Pi_n$ solve the uniform-norm approximation problem

$$\|f - Q_n\|_\infty = \min_{Q \in \Pi_n} \|f - Q\|_\infty.$$

Then, for $f \in C^{n+1}[-1, 1]$,

$$\|f - Q_n\|_\infty \leqslant \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!}, \tag{14}$$

see Meinardus [15, Theorem 60]. The bound (14) is sharp. The closeness of the bounds (9) and (14) for large m suggests that for m sufficiently large the polynomial Φ_n, given by (5), is a good approximation of f also when the error is measured in the uniform norm.

We turn to the case when $n=m-1$. Then Φ_n interpolates f at the nodes (1). A well-known difficulty arises: even for a function f analytic on $[-1, 1]$, the approximant Φ_n may oscillate with large amplitude near the endpoints of $[-1, 1]$, and the amplitude may increase with n. An analysis of this behavior, known as the *Runge phenomenon*, is presented by Runge [19], and more recently by Rivlin [18] and Li and Saff [14]. The difficulty is caused by the exponential growth with n of the norm of the interpolation operator; see [18, p. 99].

A bound analogous to (8) for Gram polynomials is shown to be valid in Section 2. This suggests that Φ_n, given by (5), approximates analytic functions f well on $[-1, 1]$ if the degree n is small enough in relation to m, so that $\|\phi_{n+1}\|_\infty$ stays bounded as n and m increase. We therefore need to study the growth of $\phi_n(x)$ as a function of m, n, and x. In Section 3 we derive a family of second order ordinary differential equations from the three-term recurrence relation for the ϕ_n. For each fixed value of $x \in [-1, 1]$, we obtain a differential equation that describes the behavior of $\phi_n(x)$ for large values of m and n. The differential equation as well as the initial conditions on the solution depend on the parameter $x \in [-1, 1]$. The solution of each initial value problem can be expressed in terms of the confluent hypergeometric function

$$F(a, c, z) := {}_1F_1(a; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n, \tag{15}$$

where $(a)_n := \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol and Γ denotes the Γ-function. Different values of x correspond to different values of the parameter a. The function (15) is also known as Kummer's function.

Section 4 shows that the solution of the initial value problem corresponding to $x=1$ dominates the solutions corresponding to $-1 \leqslant x < 1$. Therefore, it suffices to consider only the former solution when studying the growth of $\|\phi_n\|_\infty$ as m and n increase. The fact that the solution corresponding to $x=1$ dominates solutions associated with the other values of x is equivalent to the inequality

$$F\left(\frac{1-\zeta}{2}, 1, z\right) \leqslant F(1/2, 1, z) \qquad \text{for all} \quad \zeta \geqslant 0 \quad \text{and} \quad z \geqslant 0. \tag{16}$$

The proof of (16) given in Section 4 is believed to be new.

Our study of solutions to the differential equation shows that for large values of m and n, the norm $\|\phi_n\|_\infty$ is nearly invariant under changes in n and m, whenever the ratio $n/m^{1/2}$ is kept constant. Moreover, Φ_n defined by (5) is a good approximant of f in the uniform norm, provided that n is not larger than a small multiple of $m^{1/2}$, say $n \leqslant 2.5m^{1/2}$. Numerical examples that illustrate the behavior of the Gram polynomials are presented in Section 5.

The relevance of the ratio $n/m^{1/2}$ has previously been noted by Björk [5] and Zaremba [22] in their investigation of Gram polynomials. Closely related problems are also considered in [9, 10, 13, 16, 18]. Our method of investigation also can be used to analyze classes of orthogonal polynomials other than Gram polynomials.

2. GRAM POLYNOMIALS

The Gram polynomials introduced in Section 1 satisfy the three-term recurrence relation, for $1 \leqslant n < m$,

$$\phi_n(x) = 2\alpha_{n-1} x\phi_{n-1}(x) - \frac{\alpha_{n-1}}{\alpha_{n-2}}\,\phi_{n-2}(x), \tag{17}$$

$$\alpha_{n-1} := \frac{m}{n}\left(\frac{n^2 - 1/4}{m^2 - n^2}\right)^{1/2}, \tag{18}$$

with $\phi_0(x) := 1$, $\phi_{-1}(x) := 0$, and $\alpha_{-1} := 1$; see, e.g., [7, (4.4.24)–(4.4.26)].

THEOREM 2.1. *Each Gram polynomial ϕ_n, $0 \leqslant n < m$, can be written as a non-negative linear combination of Legendre polynomials p_j, $0 \leqslant j \leqslant n$. In particular,*

$$\|\phi_n\|_\infty = \phi_n(1), \qquad 0 \leqslant n < m. \tag{19}$$

Proof. The theorem can be shown directly by induction. It follows also from a more general result by Askey [2, Theorem 1]. Here we verify that the conditions of Theorem 1 in [2] are satisfied. Let $\{\phi_n^*\}_{n=0}^{m-1}$ be monic Gram polynomials associated with the bilinear form (2), and let $\{p_n^*\}_{n=0}^{\infty}$ be monic Legendre polynomials. Then

$$\phi_{n+1}^*(x) = x\phi_n^*(x) - \lambda_n \phi_{n-1}^*(x), \qquad 0 \leqslant n \leqslant m-2,$$

$$p_{n+1}^*(x) = xp_n^*(x) - \delta_n p_{n-1}^*(x), \qquad n = 0, 1, ...,$$

where

$$\lambda_n := \frac{n^2}{4n^2-1}\left(1-\frac{n^2}{m^2}\right),$$

$$\delta_n := \frac{n^2}{4n^2-1}$$

and $\phi_0^*(x) := p_0^*(x) := 1$, $\phi_{-1}^*(x) = p_{-1}^*(x) = 0$. We have to show that $\delta_k \geqslant \lambda_n > 0$ for $1 \leqslant k \leqslant n$ and $0 \leqslant n \leqslant m-2$. But δ_k decreases as $k \geqslant 1$ increases, and $\delta_n \geqslant \lambda_n > 0$. Thus, the conditions of [2, Theorem 1] are satisfied, and, therefore,

$$\phi_n^*(x) = \sum_{j=0}^{n} a_{nj} p_j^*(x) \tag{20}$$

with $a_{nj} \geqslant 0$ for all $0 \leqslant j \leqslant n$ and $0 \leqslant n < m$. The inequality (19) now follows from the representation (20) and the fact that $\|p_j^*\|_\infty = p_j^*(1)$. ∎

It follows from (19) and Theorem 1.1 that the error bound

$$\|f - \Phi_n\|_\infty \leqslant \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \|\phi_{n+1}\|_\infty \lim_{x \to \infty} (x^{n+1}/\phi_{n+1}(x)), \tag{21}$$

which is analogous to (8), is valid. Substitution of $f := \phi_{n+1}$ into (21) shows the sharpness of the bound.

3. A DIFFERENTIAL EQUATION MODEL

A differential equation is derived that approximates the three-term recurrence relation for $\phi_n(x)$. The solution of the differential equation is a function of $(n-\frac{1}{2})/m^{1/2}$. In order to derive the differential equation, we first introduce $\tau := n/m^{1/2}$. Then (18) can be written as

$$\alpha_{n-1} = (1 - \tfrac{1}{4}\tau^{-2}m^{-1})^{1/2}(1-\tau^2 m^{-1})^{-1/2}. \tag{22}$$

Let τ_0 and τ_1 be constants, such that $0 < \tau_0 < \tau_1 < \infty$, and consider n as a function of τ. We obtain from (22) that

$$\alpha_{n-1} = 1 + \tfrac{1}{2}(\tau^2 - \tfrac{1}{4}\tau^{-2})\, m^{-1} + O(m^{-2}), \qquad m \to \infty, \tag{23}$$

where the convergence in (23) is uniform for $\tau_0 \leqslant \tau \leqslant \tau_1$. Note that the bound $n \geqslant \tau_0 m^{1/2}$ implies that $n \to \infty$ as $m \to \infty$. From (22), we also obtain

$$\frac{\alpha_{n-1}}{\alpha_{n-2}} = 1 + (\tau + \tfrac{1}{4}\tau^{-3})\, m^{-3/2} + O(m^{-2}), \qquad m \to \infty, \tag{24}$$

uniformly for $\tau_0 \leqslant \tau \leqslant \tau_1$. Let $x := 1 - \zeta/m$. Then (17) can be written in the form

$$\frac{\phi_n(x) - 2\phi_{n-1}(x) + \phi_{n-2}(x)}{m^{-1}} - 2m(\alpha_{n-1} - 1)\, \phi_{n-1}(x)$$
$$- m\left(1 - \frac{\alpha_{n-1}}{\alpha_{n-2}}\right) \phi_{n-2}(x) + 2\zeta\alpha_{n-1}\phi_{n-1}(x) = 0. \tag{25}$$

Substituting (23) and (24) into (25) yields

$$\frac{\phi_n(x) - 2\phi_{n-1}(x) + \phi_{n-2}(x)}{m^{-1}}$$
$$= (\tau^2 - \tfrac{1}{4}\tau^{-2} - 2\zeta)\, \phi_{n-1}(x) - (\tau + \tfrac{1}{4}\tau^{-3})\, m^{-1/2}\phi_{n-2}(x)$$
$$+ \phi_{n-1}(x)\, O(m^{-1}) + \phi_{n-2}(x)\, O(m^{-1}), \qquad m \to \infty, \tag{26}$$

where the convergence is uniform for $\tau_0 \leqslant \tau \leqslant \tau_1$. Introduce $t := \tau - \tfrac{1}{2}\Delta\tau$, $\Delta\tau := m^{-1/2}$, and substitute

$$\phi(t) := \phi_{n-1}(x)/\sqrt{2m^{1/2}} \tag{27}$$

into (26). The change of variables from τ to t makes the $O(m^{-1/2})$-term vanish. We obtain

$$\frac{\phi(t + \Delta t) - 2\phi(t) + \phi(t - \Delta t)}{(\Delta t)^2} = (t^2 - \tfrac{1}{4}t^{-2} - 2\zeta)\, \phi(t) + O(\Delta t^2), \qquad \Delta t \to 0,$$

and, hence,

$$\frac{d^2}{dt^2}\phi(t) = (t^2 - \tfrac{1}{4}t^{-2} - 2\zeta)\, \phi(t) + O(\Delta t^2), \qquad \Delta t \to 0. \tag{28}$$

The convergence in (28) is uniform for $t_0 \leqslant t \leqslant t_1$, where t_0, t_1 are arbitrary but fixed constants, such that $0 < t_0 < t_1 < \infty$. From (28) we obtain the differential equation

$$\frac{d^2}{dt^2}\,\phi(t) = (t^2 - \tfrac{1}{4}t^{-2} - 2\zeta)\,\phi(t). \tag{29}$$

The general solution of (29) is given by

$$\phi(t) = t^{1/2}e^{-t^2/2}(A\,{}_1F_1(\tfrac{1}{2}(1-\zeta), 1, t^2) + BU(\tfrac{1}{2}(1-\zeta), 1, t^2)), \tag{30}$$

where A, B are arbitrary constants, $F = {}_1F_1$ is Kummer's function (15), and U is a linearly independent logarithmic solution to Kummer's equation; see [1, p. 504] for the definition of U. The differential equation model (28), the solution (30) with $A = 1$ and $B = 0$, and Eq. (37) below were first suggested in [8].

We are interested in studying $\|\phi_n\|_\infty = \phi_n(1)$, and therefore choose $\zeta = 0$ in (29) and (30). This value of ζ corresponds to $x = 1$. Other choices of ζ are discussed below. For $\zeta = 0$, the solution (30) simplifies to, see [1, (13.6)],

$$\phi(t) = t^{1/2}(AI_0(t^2/2) + B\pi^{-1/2}K_0(t^2/2)),$$

where I_0 and K_0 are modified Bessel functions of zeroth order of the first and second kind, respectively. We note that, see [1, Chapt. 9],

$$\begin{aligned} t^{1/2}I_0(t^2/2) &= \pi^{-1/2}t^{-1/2}e^{t^2/2}(1 + O(t^{-2})), \qquad t \to \infty,\\ t^{1/2}K_0(t^2/2) &= \pi^{1/2}t^{-1/2}e^{-t^2/2}(1 + O(t^{-2})), \qquad t \to \infty, \end{aligned}$$

which shows that $t^{1/2}I_0(t^2/2)$ is a dominating solution of (29) as t increases. Moreover,

$$\begin{aligned} t^{1/2}I_0(t^2/2) &= t^{1/2}(1 + O(t^4)), & t \to 0,\\ t^{1/2}K_0(t^2/2) &= t^{1/2}((-2\ln(t/2) + \gamma)\,I_0(t) + O(t^4)), & t \to 0, \end{aligned} \tag{31}$$

where $\gamma \approx 0.577$ denotes Euler's constant.

We turn to the initial conditions. Since $\|p_{n-1}\| = p_{n-1}(1)$, we obtain from (10) and (12), that for fixed n,

$$\begin{aligned} \phi_{n-1}(1) &= (2n+1)^{1/2} + c_{n-1}O(m^{-2})\\ &= \sqrt{2m^{1/2}}\,t^{1/2} + c_{n-1}O(m^{-2}), \qquad m \to \infty. \end{aligned} \tag{32}$$

Substituting (32) into (27) yields

$$\phi(t)=t^{1/2}(1+O(t^4)), \qquad t\to 0,$$

and in view of (31), we obtain

$$\phi(t)-t^{1/2}I_0(t^2/2)=t^{1/2}O(t^4), \qquad t\to 0, \tag{33}$$

where the power of t in the $O(t^4)$-factor cannot be increased. Thus, the function

$$\phi^{(0)}(t):=t^{1/2}I_0(t^2/2) \tag{34}$$

can be used to approximate $\phi_n(1)/\sqrt{2m^{1/2}}$ in the following way. Let $\phi(t)$ be defined by (27) with $x=1$, and select $t_0>0$ sufficiently small so that the right-hand side of (33) is small for $0\leqslant t\leqslant t_0$. Analogously to (27), define

$$\hat{\phi}_{n-1}(x):=\sqrt{2m^{1/2}}\,\phi^{(0)}(t), \qquad t:=(n-\tfrac{1}{2})/m^{1/2}, \quad t_0\leqslant t\leqslant t_1,$$

and choose m large enough so that $\hat{\phi}_{n-1}(1)$ is a good approximate solution of the difference equation (25) for $t_0\leqslant t\leqslant t_1$. Since $\phi^{(0)}$ is a dominating solution of (29), it models the behavior of the scaled polynomials $\phi_n(x)/\sqrt{2m^{1/2}}$ at $x=1$ fairly well already for modest values of m. This is illustrated by numerical examples in Section 5.

We next determine initial conditions for $\zeta>0$. For bounded $\zeta>0$ and fixed n, we obtain, by (10), that

$$\begin{aligned}\phi_{n-1}(1-\zeta/m)&=p_{n-1}(1-\zeta/m)+c_{n-1}O(m^{-2})\\ &=p_{n-1}(1)+\tilde{c}_{n-1}O(m^{-1}), \qquad m\to\infty,\end{aligned}$$

where the constant $\tilde{c}_{n-1}$ is independent of m. For $\phi(t)$ defined by (27), with $x=1-\zeta/m$, we have $\phi(t)=t^{1/2}(1+O(t^2))$, $t\to 0$. Analogously to (33), we find that

$$\phi(t)-t^{1/2}e^{-t^2/2}F(\tfrac{1}{2}(1-\zeta),1,t^2)=t^{1/2}O(t^2), \qquad t\to 0.$$

The solution of (29) that models the behavior of $\phi_{n-1}(1-\zeta/m)\sqrt{2m^{1/2}}$ for $\zeta>0$ is therefore

$$\phi^{(\zeta)}(t):=t^{1/2}e^{-t^2/2}F(\tfrac{1}{2}(1-\zeta),1,t^2). \tag{35}$$

Note that $\phi^{(\zeta)}(t) \to \phi^{(0)}(t)$ as $\zeta \to 0$. The fact that $\|\phi_n\|_\infty = \phi_n(1)$ suggests the inequality

$$\phi^{(\zeta)}(t) \leqslant \phi^{(0)}(t) \qquad \text{for all} \quad \zeta \geqslant 0, \quad t \geqslant 0, \tag{36}$$

which is equivalent with (16). We will show (36) in Section 4.

Let $\zeta := 2k+1$ for some integer $0 \leqslant k < m$. Then $x := 1 - \zeta/m$ is the node x_{m-k} defined by (1), and we obtain from (35) that the solution

$$\phi^{(\zeta)}(t) = t^{1/2} e^{-t^2/2} F(-k, 1, t^2) = t^{1/2} e^{-t^2/2} L_k(t^2) \tag{37}$$

of (29) models the behavior of $\phi_{n-1}(1-\zeta/m)/\sqrt{2m^{1/2}}$. Here $L_k(x)$ denotes a Laguerre polynomial of degree k; see [1, (22.5.54)]. The fact that $\phi^{(\zeta)}(t) \to 0$ as $t \to \infty$ agree well with the observed behavior of the polynomials ϕ_{n-1} at the nodes; see Table III of Section 5.

4. AN INEQUALITY FOR KUMMER'S FUNCTION

Inequality (19), the connection between Gram polynomials and the confluent hypergeometric function exposed in Section 3, and numerical evidence suggested the stronger inequality (16). The latter inequality was first presented as a conjecture in 1985 [3], and is also discussed in [4, p. 21]. For completeness, and because of its independent interest, we verify a generalization of this conjecture with sharp constants.

THEOREM 4.1. *For all* $\zeta \geqslant 0$, $x \geqslant 0$, *and* $c \geqslant 1/2$

$$F\left(\frac{1-\zeta}{2}, c, x\right) \leqslant F(1/2, c, x). \tag{38}$$

Moreover, $c \geqslant 1/2$ *is sharp, i.e., for* $c < 1/2$ *and* $x > 0$, *there is a* $\zeta > 0$ *such that inequality* (38) *fails.*

Proof. Suppose that $x > 0$ and $\zeta > 0$, and consider the special case $c = 1/2$. We will make use of the following classical identity, see [11, p. 1085, No. 9.211-3],

$$F(-\nu, \alpha+1, x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} e^x x^{-\alpha/2} \int_0^\infty e^{-t} t^{\nu+\alpha/2} J_\alpha(2\sqrt{xt})\, dt, \tag{39}$$

for $\alpha + \nu + 1 > 0$, where J_α is the Bessel function of order α. Using identity (39) with $\nu = (\zeta - 1)/2$ and $\alpha = -1/2$, it follows that $\alpha + \nu + 1 = \zeta/2 > 0$, and

$$J_\alpha(z) = J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z),$$

see [11, p. 977, No. 8.464-2]. Therefore,

$$
\begin{aligned}
F\left(\frac{1-\zeta}{2}, 1/2, x\right) &= e^x \frac{\Gamma(1/2)}{\Gamma(\zeta/2)} x^{1/4} \int_0^\infty e^{-t} t^{\zeta/2-3/4} J_{-1/2}(2\sqrt{xt})\,dt \\
&= e^x \frac{\Gamma(1/2)}{\Gamma(\zeta/2)} x^{1/4} \int_0^\infty e^{-t} t^{\zeta/2-3/4} \sqrt{\frac{2}{2\pi\sqrt{xt}}} \cos(2\sqrt{xt})\,dt \\
&= e^x \frac{1}{\Gamma(\zeta/2)} \int_0^\infty e^{-t} t^{\zeta/2-1} \cos(2\sqrt{xt})\,dt \\
&\leqslant e^x \frac{1}{\Gamma(\zeta/2)} \int_0^\infty e^{-t} t^{\zeta/2-1}\,dt \\
&= e^x = F(1/2, 1/2, x),
\end{aligned}
$$

which implies that

$$
F\left(\frac{1-\zeta}{2}, c, x\right) \leqslant F(1/2, c, x) \text{ for } c = 1/2 \text{ and for all } x \geqslant 0 \text{ and } \zeta \geqslant 0.
$$

Now suppose that $c > 1/2$. We wish to relate $F(a, c, x)$ to $F(a, 1/2, x)$. The identity, see [11, p. 863, No. 7.613-1],

$$
F(a, c, x) = \frac{\Gamma(c)\, x^{1-c}}{\Gamma(\gamma)\, \Gamma(c-\gamma)} \int_0^x t^{\gamma-1}(x-t)^{c-\gamma-1} F(a, \gamma, t)\,dt \qquad \text{for} \quad c > \gamma > 0,
$$

with $x > 0$, $\zeta > 0$, and $c > \gamma = 1/2$, yields

$$
\begin{aligned}
F\left(\frac{1-\zeta}{2}, c, x\right) &= \frac{\Gamma(c)\, x^{1-c}}{\Gamma(1/2)\, \Gamma(c-1/2)} \int_0^x t^{-1/2}(x-t)^{c-3/2} F\left(\frac{1-\zeta}{2}, 1/2, t\right) dt \\
&\leqslant \frac{\Gamma(c)\, x^{1-c}}{\Gamma(1/2)\, \Gamma(c-1/2)} \int_0^x t^{-1/2}(x-t)^{c-3/2} F(1/2, 1/2, t)\,dt \\
&= F(1/2, c, x), \qquad (41)
\end{aligned}
$$

where the inequality in (41) follows from (40) and the fact that

$$
\frac{\Gamma(c)\, x^{1-c}}{\Gamma(1/2)\, \Gamma(c-1/2)} t^{-1/2}(x-t)^{c-3/2} > 0
$$

for $x > t \geqslant 0$ and $c > 1/2$. This establishes that

$$F\left(\frac{1-\zeta}{2}, c, x\right) \leqslant F(1/2, c, x) \qquad \text{for all} \quad \zeta \geqslant 0, \quad x \geqslant 0, \quad \text{and} \quad c \geqslant 1/2.$$

The sharpness of $c = 1/2$ will follow from (the proof of) Theorem 4.2. ∎

Another concise version of inequality (42) is revealed when it is expressed in terms of the Whittaker functions $M_{\lambda,\mu}$, which are given by, see [11, p. 1087, No. 9.220-2],

$$M_{\lambda,\mu}(x) := x^{\mu + 1/2} e^{-x/2} F(\mu - \lambda + \tfrac{1}{2}, 1 + 2\mu, x).$$

THEOREM 4.2. *Suppose that* $\lambda \geqslant \mu \geqslant -1/4$. *Then for all* $x \geqslant 0$,

$$M_{\lambda,\mu}(x) \leqslant M_{\mu,\mu}(x). \tag{43}$$

Moreover, $\mu = -1/4$ *is sharp, i.e., for any* $\mu < -1/4$ *and* $x > 0$, *there is a* $\lambda > \mu$ *such that inequality* (43) *is invalid.*

Proof. Suppose that $-1/4 \leqslant \mu < \lambda$ and $x > 0$. For $c = 1 + 2\mu$ and $(1-\zeta)/2 = \mu - \lambda + \frac{1}{2}$ it follows that $c \geqslant 1/2$ and $\zeta = 2(\lambda - \mu) > 0$. We have

$$\begin{aligned} M_{\lambda,\mu}(x) &= x^{\mu + 1/2} e^{-x/2} F(\mu - \lambda + \tfrac{1}{2}, 1 + 2\mu, x) \\ &= x^{c/2} e^{-x/2} F\left(\frac{1-\zeta}{2}, c, x\right) \\ &\leqslant x^{c/2} e^{-x/2} F(1/2, c, x) \\ &= M_{\mu,\mu}(x), \end{aligned} \tag{44}$$

where the inequality (44) follows from (42). Therefore, inequality (43) holds for all $\lambda \geqslant \mu \geqslant -1/4$ and $x \geqslant 0$.

In order to demonstrate the sharpness of $\mu = -1/4$, we note the asymptotic relationship for large $\lambda > 0$ given by, see [11, p. 1089, No. 9.228],

$$M_{\lambda,\mu}(x) \sim \frac{\Gamma(1 + 2\mu)}{\sqrt{\pi}} \lambda^{-\mu - 1/4} x^{1/4} \cos\left(2\sqrt{\lambda x} - \mu\pi - \frac{\pi}{4}\right).$$

Now let $x_0 > 0$ and $\mu_0 \in (-1/2, -1/4)$ both be fixed. For each positive integer n, let λ_n satisfy

$$2\sqrt{\lambda_n x_0} - \mu_0 \pi - \pi/4 = 2n\pi, \qquad \text{i.e.,} \quad \lambda_n = \left(2n\pi + \mu_0\pi + \frac{\pi}{4}\right)^2 \Big/ (4x_0).$$

Since $-\mu_0 - \frac{1}{4} > 0$, it follows that $\lambda_n^{-\mu_0 - 1/4} \to \infty$ as $n \to \infty$. Thus, the expression $\Gamma(1+2\mu_0)/\sqrt{\pi} \cdot \lambda_n^{-\mu_0 - 1/4} x_0^{1/4}$ can be made arbitrarily large by choosing a sufficiently large positive integer n. In particular, there is a λ_n sufficiently large, such that

$$\begin{aligned} M_{\lambda_n, \mu_0}(x_0) &\sim \frac{\Gamma(1+2\mu_0)}{\sqrt{\pi}} \lambda_n^{-\mu_0 - 1/4} x_0^{1/4} \cos\left(2\sqrt{\lambda_n x_0} - \mu_0 \pi - \frac{\pi}{4}\right) \\ &= \frac{\Gamma(1+2\mu_0)}{\sqrt{\pi}} \lambda_n^{-\mu_0 - 1/4} x_0^{1/4} \cos(2n\pi) \\ &= \frac{\Gamma(1+2\mu_0)}{\sqrt{\pi}} \lambda_n^{-\mu_0 - 1/4} x_0^{1/4} \cdot 1 \\ &> M_{\mu_0, \mu_0}(x_0). \end{aligned}$$

Therefore, for each $x > 0$ and $\mu \in (-1/2, -1/4)$, there is a $\lambda > \mu$, such that $M_{\lambda, \mu}(x) > M_{\mu, \mu}(x)$. This proves the sharpness of $\mu = -1/4$, and, hence, the sharpness of $c = 1 + 2\mu = 1/2$ in Theorem 4.1. ∎

5. NUMERICAL EXAMPLES

The behavior of the Gram polynomials ϕ_n is displayed in three tables. The tables compare $\phi_n(x)$, for several values of x, with the function $\phi^{(\zeta)}(t)$, which is given by either (34), (35), or (37) depending on the value of ζ. Throughout this section $x := 1 - \zeta/m$ and $t := (n - \frac{1}{2})/m^{1/2}$. We use the notation $M(E)$ for the number $M \cdot 10^E$ in the tables. All computations were carried out on a VAX 11/780 computer in double precision arithmetic, i.e., with about 15 significant digits.

Table I shows the error $(\phi_{n-1}(x)/\sqrt{2m^{1/2}} - \phi^{(\zeta)}(t))/\sqrt{t}$ for $\zeta = 0$ and $\zeta = 1/2$. Columns 4 and 5 show that $\phi_{n-1}(1) > \phi_{n-1}(1 - \frac{1}{2}m)$, in agreement with our analysis. Columns 6 and 7 illustrate the convergence of the error $(\phi_{n-1}(x)/\sqrt{2m^{1/2}} - \phi(t))/\sqrt{t}$ as m increases and t is in a fixed interval. Note that the error is positive.

Table II displays the rapid growth of $\phi^{(0)}(t)$ with t. Recall that $\sqrt{2m^{1/2}}\phi^{(0)}(t)$ approximates $\phi_n(1)$ for $t = (n - \frac{1}{2})/m^{1/2}$. The fast growth of $\phi^{(0)}(t)$ with t indicates that $\phi_n(1)$ grows rapidly with $t = (n - \frac{1}{2})/m^{1/2}$. The table suggests that the choice of m and n should be such that $t = (n - \frac{1}{2})/m^{1/2} \leqslant 2.5$ in order to keep the norm $\|\phi_n\|_\infty$ modest. The norm $\|\phi_{n-1}\| = \phi_{n-1}(1)$ can be determined from Table I for such values of m and n.

Table III shows the behavior of $\phi_{n-1}(x_m)/\sqrt{2m^{1/2}}$, where the node x_m is defined by (1). Thus, $\zeta = 1$. The table shows that both $\phi^{(1)}(t)$ and $\phi_{n-1}(x_m)/\sqrt{2m^{1/2}}$ are small for large values of t.

TABLE I

Accuracy for Increasing m for t in a Fixed Interval; $x = 1 - \zeta/m$

			$\phi_{n-1}(x)/\sqrt{2m^{1/2}}$		$\left(\frac{\phi_{n-1}(x)}{\sqrt{2m^{1/2}}} - \phi^{(\zeta)}(t)\right)\Big/\sqrt{t}$	
m	$n-1$	t	$\zeta = 0$	$\zeta = \frac{1}{2}$	$\zeta = 0$	$\zeta = \frac{1}{2}$
20	1	0.34	5.80(−1)	5.65(−1)	4.61(−4)	3.37(−3)
20	5	1.23	1.29	8.67(−1)	1.22(−2)	6.53(−3)
20	10	2.35	7.08	2.48	6.05(−1)	1.99(−1)
40	1	0.24	4.87(−1)	4.81(−1)	1.15(−4)	1.62(−3)
40	5	0.87	9.68(−1)	7.97(−1)	2.06(−3)	2.53(−3)
40	10	1.66	2.01	1.02	2.42(−2)	8.98(−3)
40	15	2.45	8.29	2.81	4.02(−1)	1.30(−1)
80	1	0.12	4.10(−1)	4.07(−1)	2.87(−5)	7.97(−4)
80	5	0.61	7.92(−1)	7.19(−1)	4.57(−4)	9.77(−4)
80	10	1.17	1.22	8.50(−1)	2.53(−3)	1.48(−3)
80	15	1.73	2.19	1.06	1.52(−2)	5.51(−3)
80	20	2.29	5.65	2.04	1.07(−1)	3.53(−2)
80	21	2.41	7.17	2.48	1.60(−1)	5.21(−2)

TABLE II

Growth of $\phi^{(0)}(t) := t^{1/2} I_0(t^2/2)$

t	$\phi^{(0)}(t)$
2.0	3.22
2.5	8.57
3.0	3.03(1)
3.5	1.41(2)
4.0	8.55(2)

TABLE III

$m = 81$ and $\zeta = 1$

t	$\phi^{(1)}(t)$	$n-1$	$\phi_{n-1}(x_m)/\sqrt{2m^{1/2}}$
0.5	6.24(−1)	4	6.25(−1)
1.5	3.98(−1)	13	3.96(−1)
2.5	6.95(−2)	22	6.68(−2)
3.5	4.09(−3)	31	3.48(−3)
4.5	8.50(−5)	40	5.32(−5)
5.5	1.33(−7)	49	2.07(−7)

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REFERENCES

1. M. Abramowitz, and I. A. Stegun, "Handbook of Mathematical Functions," NBS Applied Mathematics Series, Vol. 55, 1968.
2. R. Askey, Orthogonal expansions with positive coefficients, II, *SIAM J. Math. Anal.* **2** (1971), 340–346.
3. R. W. Barnard, Survey of open problems and conjectures in complex analysis and special functions, talk presented at "Symposium on the Proof of the Bieberbach Conjecture," Purdue University, West Lafayette, (1985).
4. R. W. Barnard, Open problems and conjectures in complex analysis, *in* "Computational Methods and Function Theory" (S. Ruscheweyh, E. B. Saff, L. C. Salinas, and R. S. Varga, Eds.), Lecture Notes in Mathematics, Vol. 1435, Springer-Verlag, New York, 1990.
5. H. Björk, "Contribution to the Problem of Least Squares Approximation," Report Na 71.37, Department of Numerical Analysis, Royal Institute of Technology, Stockholm, 1971.
6. H. Brass, Error estimates for least squares approximations by polynomials, *J. Approx. Theory* **41** (1984), 345–349.
7. G. Dahlquist and Å. Björck, "Numerical Methods," Prentice Hall, Englewood Cliffs, NJ, 1974.
8. G. Dahlquist and L. Reichel, Polynomial approximation studied by a differential equation model, unpublished lecture notes, Stockholm and Moscow, August 1984.
9. H. Ehlich, Polynome zwischen Gitterpunkten, *Math. Z.* **93** (1966), 144–153.
10. H. Ehlich and K. Zeller, Schwankung von Polynomen zwischen Gitterpunkten, *Math. Z.* **86** (1964), 41–44.
11. I. S. Gradshteyn and I. M. Ryzhik, "Tables of Integrals, Series, and Products," 5th ed., Academic Press, Orlando, 1994.
12. F. B. Hildebrand, "Introduction to Numerical Analysis," Dover, New York, 1987.
13. A. Kroó, A comparison of uniform and discrete polynomial approximation, *Anal. Math.* **5** (1979), 35–49.
14. X. Li and E. B. Saff, Local convergence of Lagrange interpolation associated with equisidant nodes, *J. Approx. Theory* **78** (1994), 213–225.
15. G. Meinardus, "Approximation of Functions: Theory and Numerical Methods," Springer-Verlag, New York, 1967.
16. L. Reichel, On polynomial approximation in the uniform norm by the discrete least squares method, *BIT* **26** (1986), 349–368.
17. T. J. Rivlin, "An Introduction to the Approximation of Functions," Blaisdell, Waltham, 1966.
18. T. J. Rivlin and E. W. Cheney, A comparison of uniform approximation on an interval and a finite subset thereof, *SIAM J. Number. Anal.* **3** (1966), 311–320.
19. C. Runge, Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten, *Z. Math.* **46** (1901), 224–243.
20. G. Szegő, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc., Providence, RI, 1975.
21. M. W. Wilson, Convergence properties of discrete analogs of orthogonal polynomials, *Computing* **5** (1970), 1–5.
22. S. K. Zaremba, Some properties of polynomials orthogonal over the set $\langle 1, 2, ..., N \rangle$, *Ann. Mat. (4)* **105** (1975), 333–345.